Posets for which Verdier duality holds

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Abstract
We discuss two known sheaf-cosheaf duality theorems: Curry's for the face posets of finite regular CW complexes and Lurie's for compact Hausdorff spaces, i.e., covariant Verdier duality. We provide a uniform formulation for them and prove their generalizations. Our version of the former works over the sphere spectrum and for more general finite posets, which we characterize in terms of the Gorenstein* condition. Our version of the latter says that the stabilization of a proper separated ∞-topos is rigid in the sense of Gaitsgory. As an application, for stratified topological spaces, we clarify the relation between these two duality equivalences.

Keywords Verdier duality · Combinatorial topology · Posets · Stratification · Cosheaves · ∞-toposes

Mathematics Subject Classification Primary 18B35 · 18N60 · 55M05; Secondary 05E40 · 18B25 · 57N80

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1 Introduction

For the face poset P of a locally finite regular CW complex, Curry proved in [6, Theorem 7.7] that a canonical equivalence

$$\mathbb{D}\colon \mathrm{hD}^{\mathrm{b}}(\mathrm{Fun}(P, \mathsf{Vect})) \longrightarrow \mathrm{hD}^{\mathrm{b}}(\mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Vect})) \tag{1.1}$$

between triangulated categories[1] exists, where Vect denotes the category of vector spaces over a field. Note that the polysimplicial case was proven before by Schneider in [13, Proposition 2]. This leads to the following:

Question 1.2 Under what conditions on P do we have such a duality equivalence?

To answer this question, we must give a definition of "such a duality equivalence" so that the desired equivalence is not an extra datum but a property.

Example 1.3 Let P be the poset generated by the relations $0 \leq 1 \leq 2$ and $0 \leq 1' \leq 2$ on the set $\{0, 1, 1', 2\}$. An isomorphism $P \simeq P^{\mathrm{op}}$ gives an equivalence $\mathrm{Fun}(P, \mathsf{Sp}) \simeq \mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Sp})$, but the two isomorphisms give different equivalences. We can also use the suspension functor to get many more equivalences.

Curry's proof rules out this example, but his argument depends on a somewhat arbitrary choice of a dualizing complex.

To get some insights, let us look at a similar equivalence in general topology. Let X be a locally compact Hausdorff space. In [12, Section 5.5.5], Lurie states Verdier duality as a canonical equivalence

$$\mathbb{D}\colon \mathrm{Shv}_{\mathsf{Sp}}(X) \longrightarrow \mathrm{cShv}_{\mathsf{Sp}}(X)$$

between the ∞-categories of spectrum-valued sheaves and cosheaves. The original construction is complicated, but as we see in Sect. 6, a simpler explanation exists: We assume that X is compact for simplicity and write $p\colon X \to *$ and $d\colon X \to X \times X$ for the projection and the diagonal, respectively. Then the composites

$$\mathsf{Sp} \xrightarrow{p^*} \mathrm{Shv}_{\mathsf{Sp}}(X) \xrightarrow{d_*} \mathrm{Shv}_{\mathsf{Sp}}(X \times X) \simeq \mathrm{Shv}_{\mathsf{Sp}}(X) \otimes \mathrm{Shv}_{\mathsf{Sp}}(X), \tag{1.4}$$

$$\mathrm{Shv}_{\mathsf{Sp}}(X) \otimes \mathrm{Shv}_{\mathsf{Sp}}(X) \simeq \mathrm{Shv}_{\mathsf{Sp}}(X \times X) \xrightarrow{d^*} \mathrm{Shv}_{\mathsf{Sp}}(X) \xrightarrow{p_*} \mathsf{Sp} \tag{1.5}$$

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[1] Here h denotes the underlying triangulated category of a stable ∞-category.

constitute a duality datum for the self-duality in Pr_{st}, the symmetric monoidal ∞-category of presentable stable ∞-categories.[2] Now let us get back to our problem and take a finite poset P. It is known (see Sect. 5.2) that when X is its Alexandroff space $\mathrm{Alex}(P)$ (see Definition 5.15), we have $\mathrm{Shv}_{\mathsf{Sp}}(X) \simeq \mathrm{Fun}(P, \mathsf{Sp})$ and $\mathrm{cShv}_{\mathsf{Sp}}(X) \simeq \mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Sp})$. Moreover, in this case, both (1.4) and (1.5) are in Pr_{st}. However, these do not form a duality datum unless P is discrete:

Example 1.6 Assume $X = \mathrm{Alex}(P)$ for a finite nondiscrete poset P. Pick an element $p \in P$ that is minimal among nonminimal elements. Let $F : P \to \mathsf{Sp}$ be the extension of $\mathbf{S}$ by zero along $\{p\} \hookrightarrow P$. Then the value at p of the image of F under

$$\mathrm{Shv}_{\mathsf{Sp}}(X) \xrightarrow{\mathrm{id} \otimes (1.4)} \mathrm{Shv}_{\mathsf{Sp}}(X) \otimes \mathrm{Shv}_{\mathsf{Sp}}(X) \otimes \mathrm{Shv}_{\mathsf{Sp}}(X) \xrightarrow{(1.5) \otimes \mathrm{id}} \mathrm{Shv}_{\mathsf{Sp}}(X)$$

is given by the limit $\varprojlim(F)$, which is a coproduct of $(\#(P_{<p}) - 1)$ copies of $\Sigma^{-1}\mathbf{S}$ and never equivalent to $F(p) = \mathbf{S}$.

Therefore we give up to use (1.4) and consider if the composite

$$\mathrm{Shv}_{\mathsf{Sp}}(X) \xrightarrow{-\otimes \mathrm{Shv}_{\mathsf{Sp}}(X)} [\mathrm{Shv}_{\mathsf{Sp}}(X), \mathrm{Shv}_{\mathsf{Sp}}(X) \otimes \mathrm{Shv}_{\mathsf{Sp}}(X)] \xrightarrow{[\mathrm{id}, (1.5)]} [\mathrm{Shv}_{\mathsf{Sp}}(X), \mathsf{Sp}],$$
$$(1.7)$$

where $[-, -]$ denotes the internal mapping object in Pr_{st}, is an equivalence. Thus we reach the following:

Definition 1.8 We call a finite poset P *Verdier* if the pair $(\mathrm{Fun}(P, \mathsf{Sp}), \Gamma)$ is a commutative Frobenius algebra (cf. Sect. 2.1) in Pr_{st}, where Γ denotes the global section functor.[3]

In other words, P is Verdier if and only if (1.5) is a "perfect pairing". Our main result rephrases the Verdier property in terms of the Gorenstein* property, a concept used in combinatorial commutative algebra:

Theorem A *For a finite poset P, the following are equivalent:*

(i) *The poset P is Verdier.*
(ii) *For each $p \in P$, the full subposet $P_{<p}$ is Gorenstein* (over $\mathbf{Z}$), i.e., its geometric realization is a generalized homology sphere (see Sect. 3.2).*
(iii) *For each $p < q$ in P, the limit of $\mathbf{Z}_{[p,q]}$, the extension of the constant functor $\mathbf{Z}$ by zero along $[p, q] \hookrightarrow P$ vanishes. (Or equivalently, the limit of $\mathbf{S}_{[p,q]}$ vanishes; see Lemma 3.16.)*

The existence of an equivalence $\mathrm{Fun}(P, \mathsf{Sp}) \simeq \mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Sp})$ for P satisfying (ii) or its variant where Sp is replaced by $D(\mathbf{Z})$ may not surprise experts. What is novel is the formulation itself, with which an if-and-only-if statement becomes possible.

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[2] We prove in Sect. 6.1 that $\mathrm{Shv}_{\mathsf{Sp}}(X)$ is rigid, which is stronger than this claim.

[3] This condition is a priori different from the requirement that (1.7) is an equivalence, but it turns out to be equivalent by Lemma 2.1 as $\mathrm{Shv}_{\mathsf{Sp}}(X) \simeq \mathrm{Fun}(P, \mathsf{Sp})$ is compactly generated in this case.

Corollary 1.9 *A finite poset P is Gorenstein* if and only if $P^{\triangleright}$ is Verdier.*

Example 1.10 As proven in [4, Proposition 3.1], a finite poset P is the face poset of some regular CW complex if and only if the geometric realization of $P_{<p}$ is homeomorphic to a sphere for each p. Hence any finite face poset is Verdier. In particular, we have an equivalence $\mathrm{Fun}(P, \mathsf{Sp}) \simeq \mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Sp})$.

In light of this example, the equivalence (i) $\Leftrightarrow$ (ii) in Theorem A can be informally summarized by the following:

Slogan 1.11 A finite poset enjoys Verdier duality if and only if it is homologically CW.

Of course, there is an example that is not a face poset:

Example 1.12 Let P be the face poset of the triangulation of a homology sphere that is not a sphere. Then $P^{\triangleright}$ is Verdier, but it is not the face poset of any regular CW complex.

Remark 1.13 In this paper, we work over $\mathbf{S}$ (or $\mathbf{Z}$) for simplicity, but our argument is valid over other coefficients. For example, for a field k and a finite poset P, the functor $\varprojlim \colon \mathrm{Fun}(P, \mathrm{D}(k)) \to \mathrm{D}(k)$ makes $\mathrm{Fun}(P, \mathrm{D}(k))$ a commutative Frobenius algebra in the ∞-category of k-linear (presentable) stable ∞-categories if and only if $P_{<p}$ is Gorenstein* over k for any $p \in P$.

As a byproduct of our proof, we find the following generalization of [12, Proposition 1.2.4.3], which may be of independent interest.

Theorem B *Let P be a Gorenstein* finite poset. Then for any stable ∞-category $\mathcal{C}$, a diagram $(P^{\triangleright})^{\triangleleft} \simeq (P^{\triangleleft})^{\triangleright} \to \mathcal{C}$ is a limit if and only if it is a colimit.*

We can handle the locally finite case by a limit argument; precisely, we show the following:

Theorem C *Let P a poset such that $P_{\geq p}$ is finite and $P_{<p}$ is finite and Gorenstein* for each $p \in P$ (e.g., the face poset of a locally finite regular CW complex). Then there is a canonical equivalence*

$$\mathbb{D} \colon \mathrm{Fun}(P, \mathsf{Sp}) \longrightarrow \mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Sp}),$$

which is pointwise given by $\mathbb{D}(F) \colon p \mapsto \varprojlim_{q \in P} \mathrm{Map}(p, q) \otimes F(q)$, where $\otimes$ denotes the copower.

Remark 1.14 In Theorem C, the requirement that $P_{\geq p}$ be finite cannot be dropped: Let P be the poset given in [1, Example A.13]. Then $P_{<p}$ is Gorenstein* for any p; in fact, P is the face poset of a regular CW structure of S^{∞}. However, the assignment described in the statement does not preserve compact objects, thus does not lift to an equivalence.

Our formulation gives us more than aesthetic satisfaction. For instance, this unified view to the two duality theorems enables us to study the interaction between stratification and Verdier duality. A sample application is the following:

Theorem D *Let $f: X \to \mathrm{Alex}(P)$ be a stratification of a compact Hausdorff space, where P is a Verdier finite poset. Suppose that the inverse image $\mathrm{Shv}_{\mathsf{Sp}}(\mathrm{Alex}(P)) \to \mathrm{Shv}_{\mathsf{Sp}}(X)$ is fully faithful. Then our duality functor $\mathrm{Fun}(P, \mathsf{Sp}) \to \mathrm{Fun}(P^{\mathrm{op}}, \mathsf{Sp})$ can be canonically identified with the composite*

$$\mathrm{Shv}_{\mathsf{Sp}}(\mathrm{Alex}(P)) \xrightarrow{f^*} \mathrm{Shv}_{\mathsf{Sp}}(X) \xrightarrow{\mathbb{D}} \mathrm{cShv}_{\mathsf{Sp}}(X) \xrightarrow{f_+} \mathrm{cShv}_{\mathsf{Sp}}(\mathrm{Alex}(P)),$$

where $\mathbb{D}$ is the Verdier duality equivalence for X and f_+ is the cosheaf pushforward.

Example 1.15 In a nice situation, the (space-valued) inverse image $\mathrm{Shv}(\mathrm{Alex}(P)) \to \mathrm{Shv}(X)$ is fully faithful and its image consists of constructible sheaves; see [5, Section 3][4] for a precise statement. For example, we can show that the assumption of Theorem D is satisfied when X is a finite regular CW complex and P is its face poset.

This paper is organized as follows: We develop necessary tools on duality in Sect. 2 and on poset (co)homology in Sect. 3. Then we prove Theorem A by showing (ii) $\Rightarrow$ (iii), (iii) $\Leftrightarrow$ (i), and (i) $\Rightarrow$ (ii) in Sects. 3.3, 4.2 and 4.3, respectively. We also show Theorem B in Sect. 4.3. After that, we study its variants in Sect. 5 and in particular prove Theorem C. In Sect. 6, we study Verdier duality for locally compact Hausdorff spaces from a formal standpoint. It motivates our formulation and is used to obtain Theorem D.

Conventions

For a poset P, we write $P_\perp$ and $P_\top$ for the posets obtained by adding the least element $\perp$ and the greatest element $\top$, respectively. When we regard P as an ∞-category, these correspond to its left and right cones ($P^\lhd$ and $P^\rhd$) in [11, Notation 1.2.8.4]. We also write $P_{\perp,\top}$ for the one obtained by adding both. The empty face (or (-1)-face) is not included in our face poset, but we regard $S^{-1} = \emptyset$ as a sphere.

We use the closed symmetric monoidal structure on Pr given in [12, Section 4.8.1]. For an ∞-topos $\mathcal{X}$ and a presentable ∞-category $\mathcal{C}$, the ∞-categories of $\mathcal{C}$-valued sheaves $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X})$ and cosheaves $\mathrm{cShv}_{\mathcal{C}}(\mathcal{X})$ are identified with $\mathcal{C} \otimes \mathcal{X}$ and $[\mathcal{X}, \mathcal{C}]$, respectively. Concretely, their objects can be regarded as limit-preserving functors $\mathcal{X}^{\mathrm{op}} \to \mathcal{C}$ and colimit-preserving functors $\mathcal{X} \to \mathcal{C}$, respectively. We write $f_+ \dashv f^+$ for the pushforward-pullback adjunction for cosheaves. The global section functor, i.e., the cohomology functor, is denoted by Γ, not by $R\Gamma$.

2 General facts on duality

2.1 A useful criterion

Recall that for objects A, $A^\vee$ and a morphism $e: A^\vee \otimes A \to \mathbf{1}$ in a symmetric monoidal ∞-category, we say that e is a *counit* of a duality between A and $A^\vee$ if for any objects C and D the composite

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[4] Beware that this part contains a minor error; see Remark 5.20.

$$\mathrm{Map}(C, D \otimes A^\vee) \xrightarrow{-\otimes A} \mathrm{Map}(C \otimes A, D \otimes A^\vee \otimes A) \xrightarrow{\mathrm{Map}(C \otimes A, D \otimes e)} \mathrm{Map}(C \otimes A, D)$$

is an equivalence.

Lemma 2.1 *Let A and $A^\vee$ be objects and $e\colon A^\vee \otimes A \to \mathbf{1}$ a morphism in a closed symmetric monoidal ∞-category. If A is dualizable and the composite*

$$A^\vee \simeq [\mathbf{1}, A^\vee] \xrightarrow{-\otimes A} [A, A^\vee \otimes A] \xrightarrow{[A, e]} [A, \mathbf{1}] \tag{2.2}$$

is an equivalence, then e is a counit. Here $[-, -]$ denotes the mapping object functor.

Proof By the definition of $[-, -]$, it suffices to show that the morphism

$$C \otimes A^\vee \simeq [\mathbf{1}, C \otimes A^\vee] \xrightarrow{-\otimes A} [A, C \otimes A^\vee \otimes A] \xrightarrow{[A, C \otimes e]} [A, C]$$

is an equivalence for every C. Since A is dualizable, this morphism is equivalent to the one obtained by applying $C \otimes -$ to (2.2). □

2.2 Functorialities

For a commutative algebra A and a morphism $l\colon A \to \mathbf{1}$ (of objects) in a closed symmetric monoidal ∞-category, we can form a morphism $A \to [A, \mathbf{1}]$ as in (2.2) by letting e be the composite $l \circ m$ where $m\colon A \otimes A \to A$ is the multiplication. We discuss the (1-categorical) naturality of this assignment $(A, l) \mapsto (A \to [A, \mathbf{1}])$. Note that the pair (A, l) is called a *commutative Frobenius algebra* if e is a counit.

Proposition 2.3 *Suppose that $f\colon A \to B$ is a morphism of commutative algebras in a symmetric monoidal ∞-category and that $g\colon B \to A$ is an A-linear morphism. Then for every morphism $l\colon A \to \mathbf{1}$ and any objects C and D, there are commutative squares*

$$
\begin{array}{ccc}
\mathrm{Map}(C, D \otimes A) & \longrightarrow & \mathrm{Map}(C \otimes A, D) \\
{\scriptstyle (D \otimes f)\circ-}\downarrow & & \downarrow{\scriptstyle -\circ(C \otimes g)} \\
\mathrm{Map}(C, D \otimes B) & \longrightarrow & \mathrm{Map}(C \otimes B, D),
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Map}(C, D \otimes B) & \longrightarrow & \mathrm{Map}(C \otimes B, D) \\
{\scriptstyle (D \otimes g)\circ-}\downarrow & & \downarrow{\scriptstyle -\circ(C \otimes f)} \\
\mathrm{Map}(C, D \otimes A) & \longrightarrow & \mathrm{Map}(C \otimes A, D),
\end{array}
$$

where the horizontal morphisms are the ones associated to (A, l) and $(B, l \circ g)$, respectively.

Moreover, if the symmetric monoidal structure is closed, the same thing holds when the mapping spaces are replaced by the internal mapping objects.

Example 2.4 Let K be an ∞-category and $i\colon K_0 \hookrightarrow K$ be an inclusion of a sieve. A direct computation shows that $f = i^*\colon \mathrm{Fun}(K, \mathsf{Sp}) \to \mathrm{Fun}(K_0, \mathsf{Sp})$ and its right adjoint g satisfy the assumptions of Proposition 2.3 in $\mathrm{Pr}_{\mathrm{st}}$.

Example 2.5 Let K be an ∞-category and $j: K_1 \hookrightarrow K$ be an inclusion of a cosieve. A direct computation shows that $f = j^*: \mathrm{Fun}(K, \mathsf{S}) \to \mathrm{Fun}(K_1, \mathsf{S})$ and its left adjoint g satisfy the assumptions of Proposition 2.3 in Pr. This is a special case of Example 2.7 below.

Example 2.6 Let $p: \mathcal{Y} \to \mathcal{X}$ be a proper geometric morphism between ∞-toposes. According to Lemma 6.7, $f = p^*: \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{Y})$ and its right adjoint g satisfy the assumptions of Proposition 2.3 in $\mathsf{Pr}_{\mathsf{st}}$.

Example 2.7 Let $j: \mathcal{Y} \to \mathcal{X}$ be an étale geometric morphism between ∞-toposes. As noted in [11, Remark 6.3.5.2], $f = j^*: \mathrm{Shv}(\mathcal{X}) \to \mathrm{Shv}(\mathcal{Y})$ and its left adjoint g satisfy the assumptions of Proposition 2.3 in Pr.

Proof of Proposition 2.3 In this proof, in order to simplify the notation, we write $(-, -)$ for $\mathrm{Map}(C \otimes -, D \otimes -)$ or $[C \otimes -, D \otimes -]$ if the symmetric monoidal structure is closed.

For the first square, we construct the 2-cells in the diagram

$$
\begin{array}{ccccc}
 & (A, A \otimes A) & \longrightarrow & (A, A) & \longrightarrow & (A, \mathbf{1}) \\
 & \downarrow{\scriptstyle (g, A \otimes A)} & & \downarrow{\scriptstyle (g, A)} & & \downarrow{\scriptstyle (g, \mathbf{1})} \\
(\mathbf{1}, A) \rightrightarrows (B, A \otimes B) & \longrightarrow (B, A \otimes A) & \longrightarrow & (B, A) & \longrightarrow & (B, \mathbf{1}). \\
{\scriptstyle (\mathbf{1}, f)}\downarrow \quad {\scriptstyle (B, f \otimes B)}\downarrow & & & & & \\
(\mathbf{1}, B) \longrightarrow (B, B \otimes B) & \longrightarrow (B, B) & & & &
\end{array}
$$

We obtain the triangle and the three rectangles by naturality. We also obtain the pentagon by the linearity of g and the functoriality of $(B, -)$.

For the second square, we construct the 2-cells in the diagram

$$
\begin{array}{ccccc}
 & (B, B \otimes B) & \longrightarrow & (B, B) & \longrightarrow & (B, A) & \longrightarrow & (B, \mathbf{1}) \\
 & \downarrow{\scriptstyle (f, B \otimes B)} & & \downarrow{\scriptstyle (f, B)} & & \downarrow{\scriptstyle (f, A)} & & \downarrow{\scriptstyle (f, \mathbf{1})} \\
(\mathbf{1}, B) \rightrightarrows (A, B \otimes A) & \longrightarrow (A, B \otimes B) & \longrightarrow & (A, B) & \longrightarrow & (A, A) & \longrightarrow & (A, \mathbf{1}). \\
{\scriptstyle (\mathbf{1}, g)}\downarrow \quad {\scriptstyle (A, g \otimes A)}\downarrow & & & & & \\
(\mathbf{1}, A) \longrightarrow (A, A \otimes A) & & & & &
\end{array}
$$

We obtain the upper triangle and the four rectangles by naturality. We also obtain the lower triangle by the linearity of g and the functoriality of $(A, -)$. $\qquad\square$

We record the following obvious consequence:

Corollary 2.8 *In the situation of Proposition 2.3, assume furthermore that (A, l) is Frobenius and that $f \circ g$ is homotopic to the identity. Then $(B, l \circ g)$ is also Frobenius.*

We also have the following variant:

Lemma 2.9 *Suppose that $f: A \to B$ is a morphism of commutative algebras in a symmetric monoidal ∞-category. Then for every morphism $m: B \to \mathbf{1}$ and any objects C and D, there is a commutative square*

$$
\begin{array}{ccc}
\mathrm{Map}(C, D \otimes A) & \longrightarrow & \mathrm{Map}(C \otimes A, D) \\
\scriptstyle (D \otimes f) \circ - \downarrow & & \uparrow \scriptstyle - \circ (C \otimes f) \\
\mathrm{Map}(C, D \otimes B) & \longrightarrow & \mathrm{Map}(C \otimes B, D),
\end{array}
$$

where the horizontal morphisms are the ones associated to $(A, m \circ f)$ and (B, m), respectively.

Moreover, if the symmetric monoidal structure is closed, the same thing holds when the mapping spaces are replaced by the internal mapping objects.

Proof We use the same notation as in the proof of Proposition 2.3. We construct the 2-cells in the diagram

$$
\begin{array}{ccccccc}
(\mathbf{1}, A) & \longrightarrow & (A, A \otimes A) & \longrightarrow & (A, A) & & \\
\scriptstyle (\mathbf{1},f) \downarrow & & \scriptstyle (A,f \otimes f) \downarrow & & \scriptstyle (A,f) \downarrow & \searrow & \\
(\mathbf{1}, B) & & (A, B \otimes B) & \longrightarrow & (A, B) & \longrightarrow & (A, \mathbf{1}) \\
& \searrow & \uparrow \scriptstyle (f, B \otimes B) & & \uparrow \scriptstyle (f, B) & & \uparrow \scriptstyle (f, \mathbf{1}) \\
& & (B, B \otimes B) & \longrightarrow & (B, B) & \longrightarrow & (B, \mathbf{1}).
\end{array}
$$

We obtain the upper square since f is a morphism of commutative algebras. We also obtain the other cells by naturality. $\square$

3 Homotopy theory of posets

3.1 Poset cohomology

Definition 3.1 For a poset P, we write $|P|$ for the geometric realization (as a topological space) of its nerve and $\Delta(P)$ for its *order complex*, i.e., the abstract simplicial complex consisting of finite (nonempty) chains in P. Note that $|P|$ is canonically homeomorphic to the geometric realization of $\Delta(P)$.

In this subsection, we study how the cohomology of $|P|$ and that of P, i.e., the sheaf cohomology of $\mathrm{Fun}(P, \mathsf{S})$, are related. We first recall the following from [12, Section A.1]:

Definition 3.2 We say that an ∞-topos $\mathcal{X}$ has *constant shape* if the shape $\mathrm{Sh}\,\mathcal{X}$ is corepresentable. If $\mathcal{X}_{/X}$ has constant shape for every $X \in \mathcal{X}$, we say that $\mathcal{X}$ is *locally of constant shape*. According to [12, Proposition A.1.8], this is equivalent to the condition that the constant sheaf functor $\mathcal{S} \to \mathcal{X}$ admits a left adjoint.

Example 3.3 The presheaf ∞-topos of an ∞-category is locally of constant shape. Its shape is the image under the left adjoint of $\mathsf{S} \hookrightarrow \mathsf{Cat}_\infty$.

Example 3.4 The sheaf ∞-topos of a CW complex is locally of constant shape. Its shape is the homotopy type. In fact, any CW complex is locally of singular shape in the sense of [12, Section A.4] as any open subspace is homotopy equivalent to a CW complex.

Proposition 3.5 *If an ∞-topos $\mathcal{X}$ is locally of constant shape, for any spectrum E, the canonical morphism $[\Sigma_+^\infty \mathrm{Sh}\,\mathcal{X}, E] \to \Gamma(\mathcal{X}; E)$ is an equivalence. Here $[-, -]$ denotes the mapping spectrum.*

Proof Let $p \colon \mathcal{X} \to \mathsf{S}$ denote the projection. By assumption, p^* admits a left adjoint $p_!$. When we regard objects in $\mathrm{Shv}_{\mathsf{Sp}}(-)$ as limit-preserving functors $(-)^{\mathrm{op}} \to \mathsf{Sp}$, the spectrum-valued pushforward $p_* \colon \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X}) \to \mathrm{Shv}_{\mathsf{Sp}}(\mathsf{S}) \simeq \mathsf{Sp}$ and pullback $p^* \colon \mathsf{Sp} \simeq \mathrm{Shv}_{\mathsf{Sp}}(\mathsf{S}) \to \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$ are given as the precompositions with $(p^*)^{\mathrm{op}}$ and $(p_!)^{\mathrm{op}}$, respectively. Therefore, $\Gamma(\mathcal{X}; E) \simeq p_* p^* E$ is the value of $E \colon \mathsf{S}^{\mathrm{op}} \to \mathsf{Sp}$ at $p_! p^* * \simeq \mathrm{Sh}\,\mathcal{X}$, which is the cohomology $[\Sigma_+^\infty \mathrm{Sh}\,\mathcal{X}, E]$. $\square$

Corollary 3.6 *For a poset P and a spectrum E, we have a functorial (both in P and in E) equivalence $\Gamma(P; E) \simeq \Gamma(|P|; E)$, where E denotes the constant sheaves on the ∞-toposes $\mathrm{Fun}(P, \mathsf{S})$ and $\mathrm{Shv}(|P|)$, respectively.*

Proof This follows from Examples 3.3 and 3.4 and Proposition 3.5. $\square$

Remark 3.7 In fact, at least if $P_{\geq p}$ is finite for $p \in P$, we can construct a canonical geometric morphism $\mathrm{Shv}(|P|) \to \mathrm{Fun}(P, \mathsf{S})$ whose inverse image functor is fully faithful. This shows that we can take any functor $E \colon P \to \mathsf{Sp}$ as a coefficient in the statement of Corollary 3.6, but we do not need this generality in this paper.

3.2 Gorenstein* posets

We first recall the following notion from combinatorial commutative algebra. See [14, Chapter II] for a textbook account, which in particular explains where the name comes from.

Definition 3.8 We call an n-dimensional[5] finite abstract simplicial complex *Gorenstein** if its geometric realization is a generalized homology n-sphere, i.e., an (integral) homology n-manifold having the (integral) homology of an n-sphere.

The following definition is a variant of the definition of a Cohen–Macaulay poset given in [3, Section 3].

Definition 3.9 We call a finite poset P *Gorenstein** if for every $p < q$ in $P_{\perp,\top}$ the interval (p, q) has the (integral) homology of a sphere[6].

By definition if P is a Gorenstein* finite poset then (p, q) is Gorenstein* for every $p < q \in P_{\perp,\top}$.

Lemma 3.10 *Any maximal chain of a Gorenstein* finite poset P has the same length. In other words, $P_{\perp,\top}$ admits a rank function[7].*

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[5] Here n can be -1, so that the empty complex is Gorenstein*.

[6] We regard $S^{-1} = \emptyset$ as a sphere.

[7] A *rank function* on a finite poset P is a function $r \colon P \to \mathbf{Z}$ such that $r(q) = r(p) + 1$ if q is an immediate successor of p.

Proof This holds more generally for Cohen–Macaulay finite posets; see [3, Proposition 3.1]. ▫

These two definitions are compatible:

Proposition 3.11 *For a finite poset P, it is Gorenstein* if and only if $\Delta(P)$ is Gorenstein*.*

We omit the proof since it is a straightforward variant of [3, Proposition 3.3].

Corollary 3.12 *For a finite abstract simplicial complex, it is Gorenstein* if its underlying poset is Gorenstein*.*

We later need the following lemma, as we prefer cohomology:

Lemma 3.13 *For a finite poset, the Gorenstein* condition can be checked via cohomology instead of homology; i.e., P is Gorenstein* if and only if (p, q) has the cohomology of a sphere for $p < q$ in $P_{\perp, \top}$.*

Proof By definition P is Gorenstein* if and only if so is P^{op}. Hence the desired result follows from the self-duality of the ∞-category of perfect complexes over $\mathbf{Z}$. ▫

3.3 A vanishing result

Definition 3.14 Let P be a poset and E a spectrum. For $p \leq q$ in P, we let $E_{[p,q]} \in \mathrm{Fun}(P, \mathsf{Sp})$ denote the functor obtained from the constant functor $E \in \mathrm{Fun}([p, q], \mathsf{Sp})$ by left Kan extending along $[p, q] \hookrightarrow P_{\leq q}$ and then right Kan extending along $P_{\leq q} \hookrightarrow P$. If $E \in \mathrm{D}(\mathbf{Z})$, we use the same symbol for the element in $\mathrm{Fun}(P, \mathrm{D}(\mathbf{Z}))$ determined similarly.

Proposition 3.15 *For a Gorenstein* finite poset P, for every $p < q$ in $P_{\top}$ the cohomology $\Gamma(P_{\top}; \mathbf{Z}_{[p,q]})$ vanishes.*

We later prove the converse.

Proof Since $\Gamma(P_{\top}; \mathbf{Z}_{[p,q]}) \simeq \Gamma((P_{\top})_{\leq q}; \mathbf{Z}_{[p,q]})$ holds and $(P_{\top})_{<q}$ is also Gorenstein*, we can assume $q = \top$. By Lemma 3.10, there is a unique rank function $r \colon P_{\perp, \top} \to \mathbf{Z}$ satisfying $r(\perp) = -1$. Then $|P|$ is a generalized homology $(r(\top) - 1)$-sphere. If $r(\top) = 1$ holds, P is the discrete poset with two elements and the result can be directly checked. So we henceforth assume $r(\top) > 1$.

In what follows, we repeatedly use Corollary 3.6. Let $\mathbf{Z}_{\geq p}$ denote the left Kan extension of the constant functor with value $\mathbf{Z}$ along $P_{\geq p} \hookrightarrow P$. Then the diagram

$$
\begin{array}{ccc}
\Gamma(P_{\top}; \mathbf{Z}_{[p,\top]}) & \longrightarrow & \Gamma(P_{\top}; \mathbf{Z}) \\
\downarrow & & \downarrow{\scriptstyle f} \\
\Gamma(P; \mathbf{Z}_{\geq p}) & \xrightarrow{\ g\ } & \Gamma(P; \mathbf{Z})
\end{array}
$$

in $D(\mathbf{Z})$ obtained by restricting $\mathbf{Z}_{[p,\top]} \to \mathbf{Z}$ from $P_\top$ to P is cartesian since it induces an equivalence between the vertical fibers. As $P \to P_\top$ induces an isomorphism on $H^0(|-|; \mathbf{Z})$, we see that f induces an isomorphism on π_0. On the other hand, $\Gamma(P; \mathbf{Z}_{\geq p})$ is computed as the relative cohomology $\mathrm{fib}(\Gamma(P; \mathbf{Z}) \to \Gamma(P \backslash P_{\geq p}; \mathbf{Z}))$. By the Lefschetz duality theorem, it is the dual of $\Sigma^{r(\top)-1}\Gamma(P_{\geq p}; \mathbf{Z})$ in $D(\mathbf{Z})$, which is $\Sigma^{1-r(\top)}\mathbf{Z}$, and g induces an isomorphism on $\pi_{1-r(\top)}$ as $|P_{\geq p}|$ is connected. Therefore, f and g can be identified with the two direct summand inclusions of $\Gamma(P; \mathbf{Z}) \simeq \mathbf{Z} \oplus \Sigma^{1-r(\top)}\mathbf{Z}$, from which $\Gamma(P_\top; \mathbf{Z}_{[p,\top]}) \simeq 0$ follows. $\qquad\square$

Proof of (ii) $\Rightarrow$ (iii) **of Theorem A**   This follows from Proposition 3.15. $\qquad\square$

We note that this vanishing also holds when $\mathbf{Z}$ is replaced by $\mathbf{S}$:

Lemma 3.16 *For a finite poset P and $p \leq q$ in P, the cohomology $\Gamma(P; \mathbf{Z}_{[p,q]})$ vanishes if and only if so does $\Gamma(P; \mathbf{S}_{[p,q]})$.*

Proof Note that if a spectrum E is nonzero and bounded below, $E \otimes \mathbf{Z}$ is also nonzero; this can be seen by considering the smallest i such that $\pi_i E$ is nonzero. Hence the desired result follows from $\Gamma(P; \mathbf{Z}_{[p,q]}) \simeq \Gamma(P; \mathbf{S}_{[p,q]}) \otimes \mathbf{Z}$ and the fact that $\Gamma(P; \mathbf{S}_{[p,q]})$ is bounded below, both of which follow from the finiteness of P. $\qquad\square$

4 Verdier duality for finite posets

4.1 Recollements

We refer the reader to [12, Section A.8] for a discussion on recollements using ∞-categories. When we say $\mathcal{C}_0$ and $\mathcal{C}_1$ form a recollement, $\mathcal{C}_0$ is supposed to be the "closed" part; i.e., the $\mathcal{C}_1$-localization annihilates $\mathcal{C}_0$. We abuse terminology to say the two functors $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $\mathcal{C}_1 \hookrightarrow \mathcal{C}$ determine a recollement when $\mathcal{C}$ is a recollement of their images.

We recall the following standard fact:

Lemma 4.1 *Consider a presentable stable ∞-category $\mathcal{C}$ and suppose that $j: P_1 \hookrightarrow P$ be an upward-closed full subposet with complement $i: P_0 \hookrightarrow P$. Let i_* and j_* denote the right Kan extension along i and j and $j_!$ the left Kan extension along j. Then the following hold for the functor ∞-category $\mathrm{Fun}(P, \mathcal{C})$:*

1. *The functors i_* and j_* form a recollement.*
2. *The functors $j_!$ and i_* also form a recollement.*

Proof Both can be easily checked by using [12, Proposition A.8.20] and observing that i_* and $j_!$ are given as the extension-by-zero functors. $\qquad\square$

Lemma 4.2 *Consider a left exact functor $f: \mathcal{C} \to \mathcal{C}'$ and suppose that $\mathcal{C}$ and $\mathcal{C}'$ are recollements of $\mathcal{C}_0$ and $\mathcal{C}_1$ and $\mathcal{C}'_0$ and $\mathcal{C}'_1$, respectively. Furthermore, assume the following:*

- *The functor f restricts to define equivalences $\mathcal{C}_0 \to \mathcal{C}'_0$ and $\mathcal{C}_1 \to \mathcal{C}'_1$.*

- *The morphism $L_0' \circ f \to f \circ L_0$ obtained by the above condition is an equivalence.*

Then f itself is an equivalence.

Proof This follows from [12, Proposition A.8.14]. ☐

4.2 Duality and the vanishing condition

We prove (i) and (iii) in Theorem A are equivalent. We start with a pointwise description of $\mathbb{D}$:

Lemma 4.3 *Let K be a finite ∞-category so that $\varprojlim \colon \mathrm{Fun}(K, \mathrm{Sp}) \to \mathrm{Sp}$ is a morphism in Pr_{st}. Then the potential duality functor*

$$\mathbb{D} \colon \mathrm{Fun}(K, \mathrm{Sp}) \longrightarrow [\mathrm{Fun}(K, \mathrm{Sp}), \mathrm{Sp}] \simeq \mathrm{Fun}(K^{\mathrm{op}}, \mathrm{Sp})$$

induced by the composite $\mathrm{Fun}(K, \mathrm{Sp}) \otimes \mathrm{Fun}(K, \mathrm{Sp}) \xrightarrow{-\otimes-} \mathrm{Fun}(K, \mathrm{Sp}) \xrightarrow{\varprojlim} \mathrm{Sp}$ (cf. (2.2)) is objectwise given by

$$F \longmapsto \left(k \longmapsto \varprojlim_{l \in K} \mathrm{Map}(k, l) \otimes F(l) \right), \tag{4.4}$$

where $\otimes$ denotes the copower.

Proof By definition, $\mathbb{D}$ is the composite $\mathrm{Fun}(K, \mathrm{Sp}) \to \mathrm{Fun}(K^{\mathrm{op}} \times K \times K, \mathrm{Sp}) \to \mathrm{Fun}(K^{\mathrm{op}}, \mathrm{Sp})$, where the first and second maps are objectwise given by $F \mapsto ((k, l, m) \mapsto \mathrm{Map}(k, l) \otimes F(m))$ and $G \mapsto \left(k \mapsto \varprojlim_l G(k, l, l) \right)$, respectively. ☐

Proof of (iii) $\Rightarrow$ (i) **of Theorem A** We show by induction on $\#P$ that $\mathbb{D}_P$ is an equivalence, which is equivalent to the Verdier property by Lemma 2.1. If $P = \emptyset$, the claim is obvious. We assume $\#P > 0$ and pick a maximal element $m \in P$. Since $\{m\}$ is upward closed, by applying Lemma 4.1 to $j \colon \{m\} \hookrightarrow P$ and $i \colon P \backslash \{m\} \hookrightarrow P$, we can form two recollements, which fit into a diagram

$$\begin{array}{ccccc}
\mathrm{Fun}(P \backslash \{m\}, \mathrm{Sp}) & \xhookrightarrow{i_*} & \mathrm{Fun}(P, \mathrm{Sp}) & \xleftarrow{j_*} & \mathrm{Sp} \\
{\scriptstyle \mathbb{D}_{P \backslash \{m\}}} \downarrow & & {\scriptstyle \mathbb{D}_P} \downarrow & & \vdots \downarrow \\
\mathrm{Fun}((P \backslash \{m\})^{\mathrm{op}}, \mathrm{Sp}) & \xhookrightarrow{i_+} & \mathrm{Fun}(P^{\mathrm{op}}, \mathrm{Sp}) & \xleftarrow{j_?} & \mathrm{Sp},
\end{array} \tag{4.5}$$

where the identification $\mathrm{Fun}(\{m\}, \mathrm{Sp}) \simeq \mathrm{Sp} \simeq \mathrm{Fun}(\{m\}^{\mathrm{op}}, \mathrm{Sp})$ is made and $j_?$ denotes the right adjoint of j^+. Concretely, i_+ is given by left Kan extension and i_*, j_*, and $j_?$ are given by right Kan extensions. Proposition 2.3 applied to Example 2.4 says that the left square commutes and that the canonical morphism $i^? \circ \mathbb{D}_P \to \mathbb{D}_{P \backslash \{m\}} \circ i^*$ is an equivalence, where $i^?$ denotes the left adjoint of i_+.

We then consider applying Lemma 4.2 to conclude the proof. Since $\mathbb{D}_{P \backslash \{m\}}$ is an equivalence by our inductive hypothesis, it remains to check that $\mathbb{D}_P$ restricts to define

the dashed arrow and that it is an equivalence. Equivalently, we need to show that the composite

$$\mathsf{Sp} \xrightarrow{j_*} \mathsf{Fun}(P, \mathsf{Sp}) \xrightarrow{\mathbb{D}_P} \mathsf{Fun}(P^{\mathrm{op}}, \mathsf{Sp}) \xrightarrow{\text{restriction Fun}} (\{p\}^{\mathrm{op}}, \mathsf{Sp}) \simeq \mathsf{Sp}$$

is zero for $p \neq m$ and an equivalence for $p = m$. Note that since this functor is colimit-preserving, it is determined by its value at $\mathbf{S}$, for which we write E_p. Lemma 4.3 says that E_p is computed as $\varprojlim_{q \in P} \mathrm{Map}(p, q) \otimes (j_*\mathbf{S})(q)$.

If $p \not\leq m$, the spectrum E_p is zero as $(j_*\mathbf{S})(q) = 0$ holds for $q \not\leq m$. If $p \leq m$, the spectrum E_p is equivalent to the cohomology $\Gamma(P; \mathbf{S}_{[p,m]})$. Hence E_p is zero for $p < m$ by the assumption (iii) and Lemma 3.16. Therefore, it remains to compute $E_m \simeq \Gamma(P; \mathbf{S}_{[m,m]})$. We pick a maximal chain $p_0 < \cdots < p_r = m$ in P. For $i = 1$, ..., r, by using $\Gamma(P; \mathbf{S}_{[p_{i-1}, p_i]}) = 0$, we have

$$\Gamma(P; \mathbf{S}_{[p_i, p_i]}) \simeq \mathrm{fib}\big(\Gamma(P; \mathbf{S}_{[p_{i-1}, p_i]}) \to \Gamma(P; \mathbf{S}_{[p_{i-1}, p_{i-1}]})\big) \simeq \Sigma^{-1} \Gamma(P; \mathbf{S}_{[p_{i-1}, p_{i-1}]}).$$

Thus we have $\Gamma(P; \mathbf{S}_{[m,m]}) = \Sigma^{-r}\mathbf{S}$. Hence the dashed arrow in (4.5) is identified with the functor Σ^{-r}, which is an equivalence.   □

Proof of (i) $\Rightarrow$ (iii) **of Theorem A** We proceed by induction on $\#P$. If $P = \emptyset$, the claim is obvious. We assume $P \neq \emptyset$ and pick a maximal element m. Then $P \setminus \{m\}$ is also Verdier by Corollary 2.8 applied to Example 2.4. Hence by our inductive hypothesis, it suffices to show that $\Gamma(P; \mathbf{S}_{[p,m]})$ vanishes for any $p < m$.

We now form the diagram (4.5), but the dashed arrow already exists in this case since both $\mathbb{D}_{P \setminus \{m\}}$ and $\mathbb{D}_P$ are equivalences. As we have observed in the above proof, the existence of the dashed arrow in particular means that $\Gamma(P; \mathbf{S}_{[p,m]})$ vanishes for $p < m$, which is what we wanted to show by Lemma 3.16.   □

4.3 The Gorenstein* condition from duality

We finally complete the proof of Theorem A. The main ingredient is the following nontrivial observation:

Proposition 4.6 *If a finite poset P is Verdier, $P_{>p}$ is also Verdier for any $p \in P$.*

The proof requires the following trivial observations:

Lemma 4.7 *Let P be a finite poset. For $p \in P$, let $\mathbf{S}_{\leq p}$ denote the right Kan extension of the constant functor with value $\mathbf{S}$ along $P_{\leq p} \hookrightarrow P$. Then $\mathsf{Fun}(P, \mathsf{Sp})$ is generated by $\mathbf{S}_{\leq p}$ under colimits and shifts.*

Proof We proceed by induction on $\#P$. There is nothing to prove if $P = \emptyset$. Assume otherwise and pick a maximal element $m \in P$. Let $\mathcal{C} \subset \mathsf{Fun}(P, \mathsf{Sp})$ be the full subcategory generated by $\mathbf{S}_{\leq p}$ under colimits and shifts. The inductive hypothesis implies that $F \in \mathsf{Fun}(P, \mathsf{Sp})$ is in $\mathcal{C}$ if $F(m)$ is zero. Hence for any $F \in \mathsf{Fun}(P, \mathsf{Sp})$ the fiber of $F \to F(m) \otimes \mathbf{S}_{\leq m}$ is in $\mathcal{C}$. Since $F(m) \otimes \mathbf{S}_{\leq m}$ is also in $\mathcal{C}$ by definition we have $F \in \mathcal{C}$.   □

Lemma 4.8 *Let P be a (not necessarily finite) poset. Then $E_{[\perp,p]}\colon P_\perp \to \mathsf{Sp}$ is a limit diagram for any $p \in P$ and any $E \in \mathsf{Sp}$.*

Proof We can assume that p is the greatest element by replacing P with $P_{\leq p}$. Then the result follows from Corollary 3.6 or, more directly, the observation that now P is weakly contractible.   □

Proof of Proposition 4.6 By induction, it suffices to consider the case where p is minimal.

Let j denote the inclusion $P_{>p} \hookrightarrow P$ and $j_!\colon \mathrm{Fun}(P_{>p}, \mathsf{Sp}) \to \mathrm{Fun}(P, \mathsf{Sp})$ the left Kan extension functor. Corollary 2.8 applied to Example 2.5 says that the pair $(\mathrm{Fun}(P_{>p}, \mathsf{Sp}), \Gamma_P \circ j_!)$ is a commutative Frobenius algebra. Hence it suffices to construct an equivalence $\Gamma_P \circ j_! \simeq \Sigma^{-1} \circ \Gamma_{P_{>p}}$ in $\mathrm{Fun}(\mathrm{Fun}(P_{>p}, \mathsf{Sp}), \mathsf{Sp})$.

We write j as the composite $P_{>p} \overset{k}{\hookrightarrow} P_{\geq p} \overset{l}{\hookrightarrow} P$. Then we have a morphism $j_! \simeq l_! \circ k_! \to l_! \circ k_*$, where $(-)_!$ and $(-)_*$ denote the left and right Kan extension functors. By applying $\Gamma_P \circ -$, we obtain $\Gamma_P \circ j_! \to \Gamma_P \circ l_! \circ k_*$. Since its cofiber can be computed as

$$\Gamma_P \circ l_! \circ \mathrm{cofib}(k_! \to k_*) \simeq \Gamma_P \circ l_* \circ \mathrm{cofib}(k_! \to k_*) \simeq \Gamma_{P_{\geq p}} \circ \mathrm{cofib}(k_! \to k_*) \simeq \Gamma_{P_{>p}},$$

where we use the minimality of p, we are reduced to showing that $\Gamma_P \circ l_! \circ k_*$ is zero.

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(P_{\geq p}, \mathsf{Sp})$ spanned by the limit diagrams. We need to show that $\Gamma_P \circ l_!$ is zero on $\mathcal{C}$. We now observe that $\mathcal{C}$ is generated under colimits and shifts by $\mathbf{S}_{[p,q]}$ for $q \in P_{>p}$: First, they are indeed limit diagrams by Lemma 4.8. Then it follows from Lemma 4.7 that $\mathrm{Fun}(P_{>p}, \mathsf{Sp})$ is generated by their restrictions. Therefore, we need to show that $(\Gamma_P \circ l_!)(\mathbf{S}_{[p,q]}) \simeq \Gamma_P(\mathbf{S}_{[p,q]})$ is zero, but this follows from (iii) of Theorem A; note that we have already proven (i) $\Rightarrow$ (iii) in Sect. 4.2.   □

Proof of (i) $\Rightarrow$ (ii) **of Theorem A** Let P be a Verdier finite poset. According to Lemma 3.13, it suffices to show that (p, q) has the (integral) *cohomology* of a sphere for $p < q$ in $P_\perp$. Since we know that $P_{\leq q}$ is Verdier from (i) $\Leftrightarrow$ (iii), we can assume that q is the greatest element of P. We can also assume that $p = \perp$ by Proposition 4.6. Hence it remains to compute the cohomology of $P_{<q}$, which is the fiber of $\Gamma(P; \mathbf{Z}) \to \Gamma(P; \mathbf{Z}_{[q,q]})$. If P is a singleton, it is obviously zero. Otherwise, Lemma 4.8 says $\Gamma(P; \mathbf{Z}) \simeq \mathbf{Z}$ and the last part of the proof of (iii) $\Rightarrow$ (i) says that $\mathbf{Z}_{[q,q]}$ is some positive desuspension of $\mathbf{Z}$. Therefore, $P_{<q}$ has the cohomology of a sphere.   □

We then obtain Theorem B as a bonus:

Proof of Theorem B By the standard (stable) Yoneda argument, we can assume $\mathcal{C} = \mathsf{Sp}$. Since P^{op} is also Gorenstein*, it suffices to show that any limiting diagram $P_{\perp,\top} \to \mathsf{Sp}$ is colimiting. Then as in the proof of Proposition 4.6, it suffices to show that $\mathbf{S}_{[\perp,p]} \in \mathrm{Fun}(P_{\perp,\top}, \mathsf{Sp})$ is colimiting for $p \in P_\top$. Since this is trivial for $p = \top$ as $P_\perp$ is weakly contractible, we assume otherwise. By the self-duality of the ∞-category of finite spectra, we are reduced to showing that $\mathbf{S}_{[p,\perp]} \in \mathrm{Fun}((P_{\perp,\top})^{\mathrm{op}}, \mathsf{Sp})$ is limiting. As $p \neq \top$, this is equivalent to the vanishing of the cohomology of $\mathbf{S}_{[p,\perp]} \in \mathrm{Fun}((P_\perp)^{\mathrm{op}}, \mathsf{Sp})$, which follows from Theorem A.   □

5 Variants

In this section, we prove Theorem C and show that our duality can be regarded as a topological sheaf-cosheaf duality.

5.1 For locally finite posets

The equivalence (1.1) exists for the face poset of a locally finite regular CW complex. We extend our duality to cover that case.

Definition 5.1 We say that a poset P is *locally finite* if $P_{\geq p}$ is finite for every $p \in P$.

This terminology is justified by considering the Alexandroff topology of P (see Definition 5.15).

Definition 5.2 For a poset P, we write $\mathrm{P^{fin}}(P)$ for the poset of finite subsets. We write $\mathrm{Down}(P)$ for the poset of sieves, i.e., downward-closed full subposets. We consider the functor

$$\mathrm{P^{fin}}(P)^{\triangleright} \longrightarrow \mathrm{Down}(P) \tag{5.3}$$

given by $S \mapsto \bigcup_{s \in S} P_{\leq s}$ and $\infty \mapsto P$.

We prove that for a nice poset P, the presheaf ∞-categories of it and its opposite can be recovered from those of full subposets of the form $\bigcup_{s \in S} P_{\leq s}$ for finite S by taking colimits in Pr.

Proposition 5.4 *For any poset P and any presentable ∞-category $\mathcal{C}$, the diagram given by the composite*

$$\mathrm{P}^{fin}(P)^{\triangleright} \xrightarrow{\ (5.3)\ } \mathrm{Down}(P) \xrightarrow{\ (\mathrm{PShv}_{\mathcal{C}}(-),-_!)\ } \mathsf{Pr}$$

is colimiting.

Proof First, note that the diagram $\mathrm{P^{fin}}(P)^{\triangleright} \to \mathrm{Down}(P) \to \mathsf{Poset}$ is colimiting. Since $\mathrm{P^{fin}}(P)$ is filtered, its composite with $\mathsf{Poset} \hookrightarrow \mathsf{Cat}_{\infty}$ is also colimiting, from which the result follows. □

Proposition 5.5 *Suppose that P is a locally finite poset and $\mathcal{C}$ is a compactly generated pointed ∞-category. Then the diagram given by the composite*

$$\mathrm{P}^{fin}(P)^{\triangleright} \xrightarrow{\ (5.3)\ } \mathrm{Down}(P) \xrightarrow{\ (\mathrm{Fun}(-,\mathcal{C}),-_*)\ } \mathsf{Pr}$$

is colimiting. Here the second arrow is well defined by Lemma 5.7 below.

The proof requires several lemmas:

Lemma 5.6 *For a poset P, let $\mathrm{Down}^{fin}(P)$ be the image of $\mathrm{P}^{fin}(P)$ under (5.3). Then $\mathrm{P}^{fin}(P) \to \mathrm{Down}^{fin}(P)$ is cofinal.*

Proof This follows from Joyal's version of Quillen's theorem A and the fact that a nonempty poset having binary joins is weakly contractible. $\square$

Lemma 5.7 *Let $i\colon K_0 \hookrightarrow K$ be a sieve inclusion of ∞-categories and $\mathcal{C}$ a presentable ∞-category. Then the right Kan extension functor $i_*\colon \mathrm{Fun}(K_0, \mathcal{C}) \hookrightarrow \mathrm{Fun}(K, \mathcal{C})$ preserves weakly contractible colimits. In particular, i_* preserves colimits if $\mathcal{C}$ is pointed.*

Proof Let $F\colon J^{\triangleright} \to \mathrm{Fun}(K_0, \mathcal{C})$ be a colimit diagram where J is weakly contractible. We need to show that $i_*(F(-))(k)\colon J^{\triangleright} \to \mathcal{C}$ is colimiting for any $k \in K$. If $k \in K_0$, the diagram is equivalent to $(F(-))(k)$, which is colimiting since so is F. If $k \notin K_0$, the diagram is equivalent to the constant diagram with value $*$, which is colimiting since J is weakly contractible. $\square$

Lemma 5.8 *Let P be a poset and $\mathcal{C}$ a compactly generated ∞-category. Then any compact object of $\mathrm{Fun}(P, \mathcal{C})$ is a left Kan extension of its restriction to some finite full subposet. If P is finite, the full subcategory of compact objects is the essential image of the inclusion $\mathrm{Fun}(P, \mathcal{C}^{\omega}) \hookrightarrow \mathrm{Fun}(P, \mathcal{C})$.*

Proof These follow from [1, Corollary 2.11 and Proposition 2.8], respectively. $\square$

Lemma 5.9 *Let P be a locally finite poset and $\mathcal{C}$ a compactly generated pointed ∞-category. Then for any $P_0 \in \mathrm{Down}(P)$, the right Kan extension functor $i_*\colon \mathrm{Fun}(P_0, \mathcal{C}) \hookrightarrow \mathrm{Fun}(P, \mathcal{C})$ preserves compact objects.*

Proof Let $p \in P_0$ be an element and C a compact object of $\mathcal{C}$. Since i_* preserves (finite) colimits by Lemma 5.7, it suffices to show that $F = i_*(j(p) \otimes C)$ is compact, where j denotes the Yoneda embedding $P_0^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(P_0, \mathrm{S})$. Now we compute $F(q)$ for $q \in P$: If $q \in P_{\geq p} \cap P_0$, it is C. If $q \notin P_0$, it is final and thus initial since $\mathcal{C}$ is pointed. Otherwise, it is initial. This computation shows that $F|_{P \setminus P_{\geq p}}$ is initial, which means that F is the left Kan extension of $F|_{P_{\geq p}}$, as $P_{\geq p}$ is upward closed. This computation also shows that $F|_{P_{\geq p}}$ takes compact values, which means by Lemma 5.8 that $F|_{P_{\geq p}}$ is compact, as $P_{\geq p}$ is finite. Hence the desired result follows. $\square$

Lemma 5.10 *Let P be a locally finite poset and $\mathcal{C}$ a compactly generated pointed ∞-category. Then every compact object in $\mathrm{Fun}(P, \mathcal{C})$ is a right Kan extension of its restriction to $\bigcup_{s \in S} P_{\leq s}$ for some $S \in \mathrm{P}^{fin}(P)$.*

Proof Let F be a compact object. By Lemma 5.8, there is a finite full subposet Q such that F can be identified with the left Kan extension of $F|_Q$ along $Q \hookrightarrow P$. We take $S = \bigcup_{q \in Q} P_{\geq q}$, which is finite since P is locally finite, and consider the inclusion $i\colon P_S = \bigcup_{s \in S} P_{\leq s} \hookrightarrow P$. Since P_S contains Q, the morphism $i_! i^* F \to F$ is an equivalence. Hence it suffices to show that the composite $i_! i^* F \to F \to i_* i^* F$ is an equivalence. As its restriction to P_S is an equivalence, we consider $p \notin P_S$. Then $(i_! i^* F)(p)$ is initial since no $q \in Q$ satisfies $q \leq p$ and $(i_* i^* F)(p)$ is final since P_S is downward closed. Since $\mathcal{C}$ is pointed, the desired claim follows. $\square$

Proof of Proposition 5.5 According to Lemma 5.9, the diagram actually lands in Pr_ω, the ∞-category of compactly generated ∞-categories and functors preserving colimits and compact objects. Since the inclusion $\mathsf{Pr}_\omega \hookrightarrow \mathsf{Pr}$ preserves colimits by [11, Theorem 5.5.3.18 and Proposition 5.5.7.6], it suffices to show that its restriction $\mathrm{P}^{\mathrm{fin}}(P)^\triangleright \to \mathsf{Pr}_\omega$ is colimiting. Furthermore, since $\mathrm{P}^{\mathrm{fin}}(P)$ is filtered, it suffices to show that its composite with $(-)^\omega \colon \mathsf{Pr}_\omega \to \mathsf{Cat}_\infty$ is colimiting. Then the desired claim follows from Lemmas 5.6 and 5.10. $\qquad\square$

One might think that the desired equivalence could be immediately obtained from Propositions 5.4 and 5.5 by taking the colimit of the assignment given by

$$\mathrm{P}^{\mathrm{fin}}(P) \ni S \longmapsto \big(\mathbb{D}_{P_S} \colon \mathsf{Fun}(P_S, \mathsf{Sp}) \to \mathsf{Fun}(P_S^{\mathrm{op}}, \mathsf{Sp})\big) \in \mathsf{Fun}(\Delta^1, \mathsf{Pr}),$$

where P_S denotes $\bigcup_{s \in S} P_{\leq s}$. However, what we have proven in Sect. 2.2 is not sufficient in order to construct such a functor directly. We avoid this issue by first constructing the desired functor $\mathbb{D}$ for P:

Definition 5.11 For a locally finite poset P, we define $\Gamma_{\mathrm{cpt}} \colon \mathsf{Fun}(P, \mathsf{Sp}) \to \mathsf{Sp}$ as the colimit of the functor $\mathrm{P}^{\mathrm{fin}}(P) \to \mathsf{Fun}(\Delta^1, \mathsf{Pr})$ given by $Q \mapsto (\Gamma \colon \mathsf{Fun}(Q, \mathsf{Sp}) \to \mathsf{Sp})$. Note that the source is identified with $\mathsf{Fun}(P, \mathsf{Sp})$ by Proposition 5.5 and the target is identified with Sp by Lemma 5.6 and the fact that $\mathrm{Down}^{\mathrm{fin}}(P)$ is weakly contractible. From this, we obtain $\mathbb{D} \colon \mathsf{Fun}(P, \mathsf{Sp}) \to \mathsf{Fun}(P^{\mathrm{op}}, \mathsf{Sp})$ as in Sect. 2.2.

Now the following two results imply Theorem C:

Proposition 5.12 _Let P be a locally finite poset and $F \colon P \to \mathsf{Sp}$ a functor. If $P_{\leq p}$ is finite for each $p \in P$, the functor $\mathbb{D}(F) \colon P^{\mathrm{op}} \to \mathsf{Sp}$ is pointwise given by $p \mapsto \varprojlim_{q \in P} \mathsf{Map}(p, q) \otimes F(q)$._

Proof We fix p and vary F. Then $F \mapsto \varprojlim_{q \in P} \mathsf{Map}(p, q) \otimes F(q)$ preserves colimits by the finiteness assumption on P. Hence we can assume that F is compact. By Lemma 5.10, we can find $S \in \mathrm{P}^{\mathrm{fin}}(P)$ such that F is a right Kan extension of its restriction to $\bigcup_{s \in S} P_{\leq s}$. By replacing S with $S \cup \{p\}$, we can assume $p \in S$. Then the desired result follows from Lemma 4.3 since $\bigcup_{s \in S} P_{\leq s}$ is finite by assumption. $\qquad\square$

Theorem 5.13 _Let P be a locally finite poset. If $P_{<p}$ is finite and Gorenstein* for each $p \in P$, the pair $(\mathsf{Fun}(P, \mathsf{Sp}), \Gamma_{cpt})$ is a commutative Frobenius algebra in Pr_{st}. In particular, $\mathbb{D}$ is an equivalence._

Proof By Lemma 2.1, we only need to show that $\mathbb{D}$ is an equivalence. For $S \in \mathrm{P}^{\mathrm{fin}}(P)$, let P_S denote $\bigcup_{s \in S} P_{\leq s} \in \mathrm{Down}(P)$. We regard $\mathbb{D}$ as an object of $\mathsf{Fun}(\Delta^1, \mathsf{Pr}_{st})$ and consider the (essential) poset of subobjects $\mathrm{Sub}(\mathbb{D})$. By Proposition 2.3 applied to Example 2.4, each $S \in \mathrm{P}^{\mathrm{fin}}(P)$ determines $\mathbb{D}_{P_S} \in \mathrm{Sub}(\mathbb{D})$. Hence we obtain the morphism of posets $\mathrm{P}^{\mathrm{fin}}(P) \to \mathrm{Sub}(\mathbb{D})$. Then we consider the composite

$$\mathrm{P}^{\mathrm{fin}}(P)^\triangleright \longrightarrow \mathrm{Sub}(\mathbb{D}) \longrightarrow \mathsf{Fun}(\Delta^1, \mathsf{Pr}_{st}),$$

where we set $\infty \mapsto \mathbb{D}$ in the first arrow. This is colimiting by Propositions 5.5 and 5.4. By assumption and Theorem A, the functor $\mathbb{D}_{P_S}$ is an equivalence for $S \in \mathrm{P}^{\mathrm{fin}}(P)$. Therefore, $\mathbb{D}$ is also an equivalence. $\qquad\square$

Remark 5.14 We can define the lower shriek functor for a morphism between posets satisfying the condition of Theorem C as the functor corresponding to the cosheaf pushforward under the duality equivalences. See Remark 6.12 for the locally compact Hausdorff case.

5.2 In terms of sheaves

We explain that our duality for a poset can be interpreted as a sheaf-cosheaf duality over its Alexandroff space, which we recall as follows:

Definition 5.15 The *Alexandroff space* $\mathrm{Alex}(P)$ of a poset P is the topological space whose underlying set is that of P and whose open sets are the upward-closed subsets.

We recall the following fact, which was first proven in [1, Example A.11].

Theorem 5.16 *(Aoki) The assignment* $F \mapsto (p \mapsto F(P_{\geq p}))$ *determines the inverse image functor of a geometric morphism*

$$\mathrm{Fun}(P, \mathsf{S}) = \mathrm{PShv}(P^{\mathrm{op}}) \longrightarrow \mathrm{Shv}(\mathrm{Alex}(P)). \tag{5.17}$$

This identifies $\mathrm{Shv}(\mathrm{Alex}(P))$ *as the bounded reflection of* $\mathrm{PShv}(P^{\mathrm{op}})$ *and* $\mathrm{PShv}(P^{\mathrm{op}})$ *as the hypercompletion of* $\mathrm{Shv}(\mathrm{Alex}(P))$.

Note that this geometric morphism is not an equivalence in general; see [1, Example A.13]. However, it is an equivalence in the situation we are interested in:

Proposition 5.18 *If* P *is a locally finite poset,* (5.17) *is an equivalence.*

Proof According to [1, Example A.12], this is true for finite posets. By Theorem 5.16, the morphism (5.17) is an equivalence if and only if $\mathrm{Shv}(\mathrm{Alex}(P))$ is hypercomplete. Since $\mathrm{Shv}(\mathrm{Alex}(P))$ can be written as a colimit of $\mathrm{Shv}(\mathrm{Alex}(P_{\geq p_1} \cap \cdots \cap P_{\geq p_n}))$ for $p_1, \ldots, p_n \in P$ and $n \geq 1$ in the ∞-category of ∞-toposes, $\mathrm{Shv}(\mathrm{Alex}(P))$ is hypercomplete when P is locally finite. $\square$

Remark 5.19 Note that by using [2, Corollary 2.6] instead of [1, Example A.12] in the proof, we can obtain this result for a wider class of posets.

Remark 5.20 It is a consequence of [5, Theorem 3.4] that the morphism (5.17) is an equivalence for a poset satisfying the ascending chain condition. However, they use the "geometric morphism" $\mathrm{Shv}(X) \to \mathrm{PShv}(P^{\mathrm{op}})$ constructed in [5, page 27] for a stratification $X \to \mathrm{Alex}(P)$, which is not geometric in general; the trivial stratification on $\mathrm{Alex}(P)$ for the poset P in [1, Example A.13] gives a counterexample. Nevertheless, when P is locally finite, Proposition 5.18 shows that the morphism is indeed geometric.

Hence Theorem C, which we have seen in Sect. 5.1, says the following:

Theorem 5.21 *Let* P *be a locally finite poset such that* $P_{<p}$ *is finite and Gorenstein* for each* $p \in P$. *Then there is a canonical equivalence*

$$\mathbb{D}\colon \mathrm{Shv}_{\mathsf{Sp}}(\mathrm{Alex}(P)) \longrightarrow \mathrm{cShv}_{\mathsf{Sp}}(\mathrm{Alex}(P)).$$

6 Verdier duality for proper separated ∞-toposes

The sheaf-cosheaf duality for locally compact Hausdorff spaces, which is often called covariant Verdier duality, was studied in [12, Section 5.5.5]. In this section, we first prove its generalization using more abstract methods. Then we prove Theorem D using our formulation. In future work, we will study a relative variant.

6.1 Proper separated ∞-toposes

Following [10, C2.4.16], we say that a geometric morphism is *Beck–Chevalley* if any pullback satisfies the Beck–Chevalley condition; i.e., the (unstable) proper base change theorem holds. Recall that in [11, Section 7.3.1] a geometric morphism is called *proper* if its arbitrary base change is Beck–Chevalley.

Definition 6.1 An ∞-topos $\mathcal{X}$ is called *separated* if its diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is proper.

Remark 6.2 Consider a geometric morphism $\mathcal{Y} \to \mathcal{X}$ between n-toposes. If the geometric morphism $\mathrm{Shv}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{X})$ between ∞-toposes is proper, its arbitrary base change is Beck–Chevalley in the $(n + 1)$-category of n-toposes, but not vice versa. This is why in 1-topos theory we usually call a geometric morphism *tidy* when its arbitrary base change is Beck–Chevalley in the 2-category of 1-toposes. The same remark applies to the notion of separatedness.

However, the following is proven in [11, Theorem 7.3.1.16]:

Example 6.3 [Lurie] The sheaf ∞-topos of a compact Hausdorff space is proper and separated.

We recall the following notion, which was introduced in [7, Appendix D]:

Definition 6.4 [Gaitsgory] A presentably symmetric monoidal stable ∞-category $\mathcal{C}$ is called *rigid* if the unit $u\colon \mathsf{Sp} \to \mathcal{C}$ admits a colimit-preserving right adjoint and the multiplication $m\colon \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ admits a $\mathcal{C} \otimes \mathcal{C}$-linear[8] right adjoint.

If $\mathcal{C}$ is rigid, it is easy to see that $u^{\mathrm{R}} \circ m$ and $m^{\mathrm{R}} \circ u$ constitute a duality datum in Pr, where $-^{\mathrm{R}}$ indicates the right adjoint. In particular, $(\mathcal{C}, u^{\mathrm{R}} \circ m)$ is a commutative Frobenius algebra.

Theorem 6.5 *If $\mathcal{X}$ is a proper separated ∞-topos, then $\mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$ is rigid.*

Corollary 6.6 *The pair $(\mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X}), \Gamma)$ is a commutative Frobenius algebra in Pr_{st} for any proper separated ∞-topos $\mathcal{X}$.*

Proof of Theorem 6.5 According to [12, Example 4.8.1.19], the binary product of ∞-toposes can be computed as their tensor product in Pr. Hence the result follows from Lemma 6.7 below. □

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[8] Here the colimit-preserving property is included in the definition of linearity.

Lemma 6.7 *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper morphism of ∞-toposes, then $f^* : \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$ $\to \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{Y})$ admits a $\mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$-linear right adjoint.*

Proof According to [11, Remark 7.3.1.5], the direct image functor $\mathcal{Y} \to \mathcal{X}$ preserves filtered colimits. Hence $f_* : \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$ preserves colimits. Now we consider the diagram

$$
\begin{array}{ccccc}
\mathcal{Y} & \xrightarrow{\mathrm{graph}} & \mathcal{Y} \times \mathcal{X} & \xrightarrow{\mathrm{pr}_1} & \mathcal{Y} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \times \mathrm{id}} & & \downarrow{\scriptstyle f} \\
\mathcal{X} & \xrightarrow{\mathrm{diagonal}} & \mathcal{X} \times \mathcal{X} & \xrightarrow{\mathrm{pr}_1} & \mathcal{X}
\end{array}
$$

in the ∞-category of ∞-toposes. Since the right and outer squares are cartesian, so is the left one. According to [12, Example 4.8.1.19], the binary product of ∞-toposes can be computed as their tensor product in Pr. Therefore, since $f \times \mathrm{id}$ is Beck–Chevalley, for any $F \in \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$ and $G \in \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{Y})$ the canonical morphism $f_* G \otimes F \to f_*(G \otimes f^* F)$ is an equivalence. □

6.2 The locally compact case

The following result is derived from Corollary 6.6 by using Corollary 2.8 applied to Example 2.7:

Theorem 6.8 *Let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ be an open subtopos of a proper separated ∞-topos. Then the pair $(\mathrm{Shv}_{\mathsf{Sp}}(\mathcal{U}), \Gamma_{\mathcal{X}} \circ j_!)$ is a commutative Frobenius algebra in Pr_{st}.*

Here the composite $\Gamma_{\mathcal{X}} \circ j_!$ depends on j, not only on $\mathcal{U}$, but there is a canonical choice for locally compact spaces:

Definition 6.9 Let X be a locally compact Hausdorff space. We define the *global section with compact support* Γ_{cpt} as the composite $p_* \circ j_!$ where $j : X \hookrightarrow X_\infty$ is the inclusion to its one-point compactification and $p : X_\infty \to *$ is the projection. Then Theorem 6.8 says that $(\mathrm{Shv}_{\mathsf{Sp}}(X), \Gamma_{\mathrm{cpt}})$ is a commutative Frobenius algebra. We let $\mathbb{D} : \mathrm{Shv}(X) \to \mathrm{cShv}(X)$ denote the associated equivalence (cf. Sect. 2.2).

Remark 6.10 One can prove Verdier duality for locally compact Hausdorff spaces by a similar method to the one we have used in Sect. 5.1 for locally finite posets: Namely, the sheaf and cosheaf ∞-categories of a locally compact Hausdorff space can be written as colimits in Pr as those of compact subspaces. We leave the details to the interested reader.

We give its objectwise description to justify calling our functor "Verdier duality":

Proposition 6.11 *For a locally compact Hausdorff space X and a spectrum-valued sheaf $F \in \mathrm{Shv}_{\mathsf{Sp}}(X)$, the cosheaf $\mathbb{D}(F)$ is pointwise given by $U \mapsto \varinjlim_{K \subset U} \mathrm{fib}(F(X) \to F(X \backslash K))$, where K runs over compact subsets.*

Note that Lurie's equivalence also has this pointwise formula; see [12, Proposition 5.5.5.10].

Proof First suppose that X is compact. Let j denote the inclusion $U \hookrightarrow X$. By definition, $\mathbb{D}(F)(U)$ is the global section of $(j_! \mathbf{S}_U) \otimes F$. Let i denote the inclusion $X \backslash U \hookrightarrow X$. Then by recollement, $\mathbb{D}(F)(U)$ is equivalent to the global section of $\mathrm{fib}(F \to i_* i^* F)$. Hence it is written as $\varinjlim_{V \supset X \backslash U} \mathrm{fib}(F(X) \to F(V))$, where V runs over open subsets. As X is compact, this coincides with the desired description.

We proceed to the general case. Let $j \colon X \hookrightarrow X_\infty$ denote the inclusion to the one-point compactification and i the inclusion of the point at infinity. Proposition 2.3 applied to Example 2.7 says $j_+ \circ \mathbb{D}_X \simeq \mathbb{D}_{X_\infty} \circ j_!$. Hence $\mathbb{D}_X(F)(U)$ can be computed as

$$(j_+ \circ \mathbb{D}_X)(F)(U) \simeq (\mathbb{D}_{X_\infty} \circ j_!)(F)(U) \simeq \varinjlim_{K \subset U} \mathrm{fib}\big((j_! F)(X_\infty) \to (j_! F)(X_\infty \setminus K)\big),$$

where we use the compact case. By recollement, the desired result follows from the vanishing of $\mathrm{fib}((i_* i^* j_* F)(X_\infty) \to (i_* i^* j_* F)(X_\infty \backslash K))$ for each K, which follows from $K \subset X$. $\qquad\square$

Remark 6.12 Let $f \colon Y \to X$ be a continuous map between locally compact Hausdorff spaces. As in [8, Remark 9.4.6], we can define the lower shriek functor $f_!$ as the composite $(\mathbb{D}_X)^{-1} \circ f_+ \circ \mathbb{D}_Y$. One could check its standard properties by applying Proposition 2.3 to Examples 2.6 and 2.7. To describe further functorial properties of this construction, one could use the technology presented in [9, Chapter 7]. However, beware that it is built on unproven results in $(\infty, 2)$-category theory.

6.3 Application: Verdier duality and stratification

We prove the following generalization of Theorem D:

Theorem 6.13 *Let P be a finite poset and $\mathcal{X} \to \mathrm{Shv}(\mathrm{Alex}(P))$ a geometric morphism. Suppose that P is Verdier, that $\mathcal{X}$ is proper and separated, and that the spectrum-valued inverse image $f^* \colon \mathrm{Shv}_{\mathsf{Sp}}(\mathrm{Alex}(P)) \to \mathrm{Shv}_{\mathsf{Sp}}(\mathcal{X})$ is fully faithful. Then we have $\mathbb{D}_P \simeq f_+ \circ \mathbb{D}_{\mathcal{X}} \circ f^*$.*

Remark 6.14 The assumption is satisfied when the *space-valued* inverse image $\mathrm{Shv}(\mathrm{Alex}(P)) \to \mathcal{X}$ is fully faithful: This can be seen by considering objects of $\mathrm{Shv}_{\mathsf{Sp}}(-)$ as left exact functors $(\mathsf{Sp}^\omega)^{\mathrm{op}} \to -$.

Proof of Theorem 6.13 We have $\Gamma_{\mathrm{Alex}(P)} \simeq \Gamma_{\mathrm{Alex}(P)} \circ f_* \circ f^* \simeq \Gamma_X \circ f^*$. Hence the desired result follows from Lemma 2.9. $\qquad\square$

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